# On Combinations of Concepts: CHSH Inequalities Do Not Tell Us Much in the Absence of Marginal Selectivity

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## Abstract

This commentary is confined to the issue of using CHSH inequalities of quantum physics to analyze concept combinations. The point made is that these inequalities are not informative unless outcome probabilities satisfy an invariance property known as marginal selectivity. In quantum physics marginal selectivity is ensured by the impossibility of causal relations between events separated by space-like intervals. No similar constraints exist in behavioral applications.

KEYWORDS: CHSH inequality; causal communication constraint; concept combinations; EPR paradigm; marginal selectivity; selective influences.

## 1. ABSTRACT ANALOGY BETWEEN EPR/B PARADIGM AND CONCEPT COMBINATIONS

The CHSH inequality (Clauser, Horne, Shimony, & Holt, 1969) can be tested for any system with two binary inputs and two binary outputs.

In Aerts, Gabora, and Sozzo's (2012) experiments, the inputs can be presented as two requests: "choose an animal" and "choose an animal sound." The first input has two variants, or values: one is $a$ = "choose between Horse and Bear," the other $a'$ = "choose between Tiger and Cat." Input "choose a sound" also has two values: $b$ = "choose between Growls and Whinnies" and $b'$ = "choose between Snorts and Meows." The outputs are denoted $A$ and $B$, with or without primes to match the treatment for which they are recorded. E.g., the output pair $(A, B')$ denotes two choices made in response to $(a, b')$.[1] Each output has two possible values encoded as $+1$ and $-1$ according as the first or the second of the two options presented is chosen. Thus, $A = -1, B' = -1$ (or $A_{-1}, B'_{-1}$ for short) denotes "Bear Meows." By repeatedly presenting the four treatments one estimates the joint probabilities $P(A_i, B_j), P(A_i, B'_j), P(A'_i, B_j), P(A'_i, B'_j)$, where $i$ and $j$ stand for $+1$ or $-1$, and $X_i, Y_j$ abbreviates $X = i, Y = j$.

In the simplest Bohmian version of the Einstein-Podolsky-Rosen paradigm (EPR/B), the system consists of two $1/2-$spin entangled particles, "Alice's" and "Bob's," and the inputs are detectors set for measuring their spins along specified axes. To emphasize the analogy, the two Alice's inputs can be presented as "requests to choose" between "spin up" and "spin down" along axis $\mathbf{a}$ (input $a$) or along axis $\mathbf{a}'$ (input $a'$); Bob's inputs $b$ and $b'$ (corresponding to axes $\mathbf{b}$ and $\mathbf{b}'$) are defined analogously. With outputs $A, A', B, B'$ and their values $+1$ and $-1$ defined as above, $A_1, B'_{-1}$ means that Alice recorded "spin up" along axis $\mathbf{a}$, and Bob recorded "spin down" along axis $\mathbf{b}'$.[2]

## 2. CHSH INEQUALITIES AND MARGINAL SELECTIVITY

For either of the situations just mentioned the following statement is (trivially) true.

**Proposition 1.** *For any set of joint probabilities* $P(X_i, Y_j)$ *(*$X \in \{A, A'\}, Y \in \{B, B'\}, i, j \in \{1, -1\}$*), one can find a random variable* $\omega$ *and eight functions* $f_{xy}, g_{xy}$

*($x \in \{a, a'\}, y \in \{b, b'\}$), such that*

$$\begin{aligned} (A, B) &\sim (f_{ab}(\omega), g_{ab}(\omega)), \quad (A, B') \sim (f_{ab'}(\omega), g_{ab'}(\omega)), \\ (A', B) &\sim (f_{a'b}(\omega), g_{a'b}(\omega)), \; (A', B') \sim (f_{a'b'}(\omega), g_{a'b'}(\omega)), \end{aligned} \tag{1}$$

*where $\sim$ stands for "is distributed as."*

Here, $\omega$ can be viewed as the totality of all "hidden variables" that can affect particle spins, or choices of animals and sounds. Mathematically, however, $\omega$ never has to be more complex than a discrete random variable with $2^8$ values.[3]

Both in quantum physics and behavioral sciences it is of great interest whether this universally true proposition can hold in a more restricted form, as follows.

**Proposition 2.** *One can find a random variable $\lambda$ and four functions $F_a, F_{a'}, G_b, G_{b'}$, such that*

$$\begin{aligned} (A, B) &\sim (F_a(\lambda), G_b(\lambda)), \quad (A, B') \sim (F_a(\lambda), G_{b'}(\lambda)), \\ (A', B) &\sim (F_{a'}(\lambda), G_b(\lambda)), \; (A', B') \sim (F_{a'}(\lambda), G_{b'}(\lambda)). \end{aligned} \tag{2}$$

In quantum physics this hypothesis amounts to the possibility of a classical explanation for the EPR/B data, with all inputs acting locally and $\lambda$ designating "hidden variables" ($\lambda$, however, if exists, need not be more than a 16-valued random variable)[4]. In psychology Dzhafarov (2003) introduced Proposition 2 as a definition for *selectiveness of influences* (exerted by inputs on outputs).

An immediate consequence of Proposition 2 is *marginal selectivity* (Dzhafarov, 2003; Townsend & Schweickert, 1983):

$$\begin{aligned} A &\sim F_a(\lambda), \; A' \sim F_{a'}(\lambda), \\ B &\sim G_b(\lambda), \; B' \sim G_{b'}(\lambda). \end{aligned} \tag{3}$$

It can be written as a constraint on joint probabilities: with $X \in \{A, A'\}, Y \in \{B, B'\}$,

$$\begin{aligned} P(X_1, B_1) + P(X_1, B_{-1}) &= P(X_1, B'_1) + P\left(X_1, B'_{-1}\right), \\ P(A_1, Y_1) + P(A_{-1}, Y_1) &= P(A'_1, Y_1) + P\left(A'_{-1}, Y_1\right). \end{aligned} \tag{4}$$

In quantum physics marginal selectivity is known under other names, such as causal communication constraint (see Cereceda, 2000). It is trivially satisfied if one assumes that the entangled particles are separated by a space-like interval (i.e., either of them may precede the other in time from the vantage point of an appropriately moving observer).

The CHSH inequality is another consequence of Proposition 2:

$$\Gamma \leq 2, \tag{5}$$

where, with E denoting expectation,

$$\Gamma = \max\left\{\pm \mathrm{E}\left[AB\right] \pm \mathrm{E}\left[AB'\right] \pm \mathrm{E}\left[A'B\right] \pm \mathrm{E}\left[\mathrm{A'B'}\right] : \text{ number of } + \text{ signs is odd}\right\}. \tag{6}$$

The inequality does not, however, imply Proposition 2, as we see in the following imaginary situation:

| | | $B$: $+1$ | $B$: $-1$ | | $B'$: $+1$ | $B'$: $-1$ | |
|---|---|---|---|---|---|---|---|
| $A$ | $+1$ | .25 | .25 | .5 | .25 | .5 | .75 |
| $A$ | $-1$ | .25 | .25 | .5 | .0 | .25 | .25 |
| | | .5 | .5 | | .25 | .75 | |

| | | $B$: $+1$ | $B$: $-1$ | | $B'$: $+1$ | $B'$: $-1$ | |
|---|---|---|---|---|---|---|---|
| $A'$ | $+1$ | .25 | .35 | .6 | .25 | .45 | .7 |
| $A'$ | $-1$ | .15 | .25 | .4 | .05 | .25 | .3 |
| | | .4 | .6 | | .3 | .7 | |

The CHSH inequality is satisfied ($\Gamma = 0$), but marginal selectivity is violated, ruling out Proposition 2. (Decimal fractions are probabilities: e.g., $P\left(A'_1, B'_{-1}\right) = 0.45$, $P\left(A'_1\right) = .7$, $P\left(B'_{-1}\right) = 0.7$.)

It is, of course, also possible that marginal selectivity is satisfied but $\Gamma$ exceeds 2. This is the focal fact in the EPR/B experiments. In the following pattern of probabilities $\Gamma = 4$ (which is the largest possible value), violating both (5) and its quantum version, with $2\sqrt{2}$ substituting for 2.

**A, B**

| | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | .5 | 0 | .5 |
| $-1$ | 0 | .5 | .5 |
| | .5 | .5 | |

**A, B′**

| | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | .5 | 0 | .5 |
| $-1$ | 0 | .5 | .5 |
| | .5 | .5 | |

**A′, B**

| | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | .5 | 0 | .5 |
| $-1$ | 0 | .5 | .5 |
| | .5 | .5 | |

**A′, B′**

| | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | 0 | .5 | .5 |
| $-1$ | .5 | 0 | .5 |
| | .5 | .5 | |

The CHSH inequality is violated ($\Gamma = 4$) with marginal selectivity satisfied.

We see that the two consequences of Proposition 2, the CHSH inequality and marginal selectivity, are logically independent. Together, however, they form a criterion for Proposition 2.

**Theorem 1.** *Proposition 2 is equivalent to the conjunction of the CHSH inequality (5) and marginal selectivity (4).*

This was first proved by Fine (1981), but with (4) implied rather than stipulated. That it was implied is apparent from the notation used. Fine replaces (5) with the four double-inequalities[5]

$$\begin{aligned}
-1 &\leq P(A_1, B_1) + P(A'_1, B_1) + P(A'_1, B'{}_1) - P(A_1, B'_1) - P(A'_1) - P(B_1) \leq 0, \\
-1 &\leq P(A_1, B'_1) + P(A'_1, B'{}_1) + P(A'_1, B_1) - P(A_1, B_1) - P(A'_1) - P(B'{}_1) \leq 0, \\
-1 &\leq P(A'_1, B_1) + P(A_1, B_1) + P(A_1, B'_1) - P(A'_1, B'{}_1) - P(A_1) - P(B_1) \leq 0, \\
-1 &\leq P(A'_1, B'{}_1) + P(A_1, B'_1) + P(A_1, B_1) - P(A'_1, B_1) - P(A_1) - P(B'{}_1) \leq 0.
\end{aligned} \tag{7}$$

The use, say, of the marginal probability $P(A_1)$ with no reference to value of the second input amounts to accepting (4) with $X = A$.[6] If marginal selectivity is violated, (7) is simply inapplicable.

## 3. WHEN MARGINAL SELECTIVITY IS VIOLATED

Consider now the experimental results reported in Aerts et al. (2012).

$A$, $B$:

| | Growls | Whinnies | |
|---|---|---|---|
| Horse | .049 | .630 | .679 |
| Bear | .259 | .062 | .321 |
| | .308 | .692 | |

$A$, $B'$:

| | Snorts | Meows | |
|---|---|---|---|
| Horse | .593 | .025 | .618 |
| Bear | .296 | .086 | .382 |
| | .889 | .111 | |

$A'$, $B$:

| | Growls | Whinnies | |
|---|---|---|---|
| Tiger | .778 | .086 | .864 |
| Cat | .086 | .049 | .135 |
| | .864 | .135 | |

$A'$, $B'$:

| | Snorts | Meows | |
|---|---|---|---|
| Tiger | .148 | .086 | .234 |
| Cat | .099 | .667 | .766 |
| | .247 | .753 | |

Probability estimates from Table 1 of Aerts et al. (2012), $n$/treatment = 81. Marginal selectivity is violated, the CHSH inequality is violated too.

Marginal selectivity here is violated in all four cases. Thus,

$$0.135 = P\left(Cat, Growls\right) + P\left(Cat, Whinnies\right) \neq P\left(Cat, Snorts\right) + P\left(Cat, Meows\right) = 0.766.$$

The difference being both large and statistically significant, we conclude that Proposition 1 in this experiment does not reduce to Proposition 2. In other words, the choice between animals is influenced not only by animal options but also by sound options; and analogously for the choice between sounds. The CHSH inequality here is violated too, $\Gamma = 2.420$, but this hardly adds insights to the rejection of Proposition 2.

To emphasize the issue of selective influences, consider the two probability patterns below, with $\varepsilon$ a number very close to zero. In the first pattern, as $\varepsilon$ increases from $\leq 0$ to $> 0$ a dramatic change occurs: representability (2) ceases to exist. $\Gamma$ is precisely the same in the next pattern. But as $\varepsilon$ reaches and then exceeds 0, nothing of significance seems happening, at least with respect to Proposition 2.

**$A \times B$**

| $A \backslash B$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.375+\varepsilon$ | $.125-\varepsilon$ | $.5$ |
| $-1$ | $.125-\varepsilon$ | $.375+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

**$A \times B'$**

| $A \backslash B'$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.375+\varepsilon$ | $.125-\varepsilon$ | $.5$ |
| $-1$ | $.125-\varepsilon$ | $.375+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

**$A' \times B$**

| $A' \backslash B$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.375+\varepsilon$ | $.125-\varepsilon$ | $.5$ |
| $-1$ | $.125-\varepsilon$ | $.375+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

**$A' \times B'$**

| $A' \backslash B'$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.125+\varepsilon$ | $.375-\varepsilon$ | $.5$ |
| $-1$ | $.375-\varepsilon$ | $.125+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

$\Gamma = 2 + 8\varepsilon$, with marginal selectivity satisfied.

**$A \times B$**

| $A \backslash B$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.365+\varepsilon$ | $.125-\varepsilon$ | $.49$ |
| $-1$ | $.145-\varepsilon$ | $.365+\varepsilon$ | $.51$ |
| | $.5$ | $.5$ | |

**$A \times B'$**

| $A \backslash B'$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.375+\varepsilon$ | $.125-\varepsilon$ | $.5$ |
| $-1$ | $.125-\varepsilon$ | $.375+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

**$A' \times B$**

| $A' \backslash B$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.375+\varepsilon$ | $.125-\varepsilon$ | $.5$ |
| $-1$ | $.125-\varepsilon$ | $.375+\varepsilon$ | $.5$ |
| | $.5$ | $.5$ | |

**$A' \times B'$**

| $A' \backslash B'$ | $+1$ | $-1$ | |
|---|---|---|---|
| $+1$ | $.125+\varepsilon$ | $.385-\varepsilon$ | $.51$ |
| $-1$ | $.385-\varepsilon$ | $.105+\varepsilon$ | $.49$ |
| | $.51$ | $.49$ | |

$\Gamma = 2 + 8\varepsilon$, with marginal selectivity violated.

Perhaps this is not the end of the story. There is a huge gap between representations (1) and (2), and a systematic theory is needed to study intermediate cases. We recently began this study (Dzhafarov & Kujala, 2012c), confining it, however, to the cases with marginal selectivity satisfied. It is conceivable that the situations where it is violated could be shown within the framework of a general theory to be structurally different depending on the value of $\Gamma$. This would impart a diagnostic value to findings like those reported in Aerts et al. (2012).

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## Notes

[1]We keep our notation close to Aerts et al.'s (2012).

[2]See Dzhafarov and Kujala (2012a, b) for a detailed discussion of quantum-behavioral conceptual parallels.

[3]This follows from an extended version of the Joint Distribution Criterion (Dzhafarov & Kujala, 2012c).

[4]This follows from the Joint Distribution Criterion, as formulated in Fine (1982). See Dzhafarov & Kujala (2012a).

[5]Clauser and Horne (1974) use the same form, but only as a necessary condition for Proposition 2.

[6]A proof with explicit derivation of (4) and (5) obtains as a special case of the Linear Feasibility Criterion described in Dzhafarov and Kujala (2012a).

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